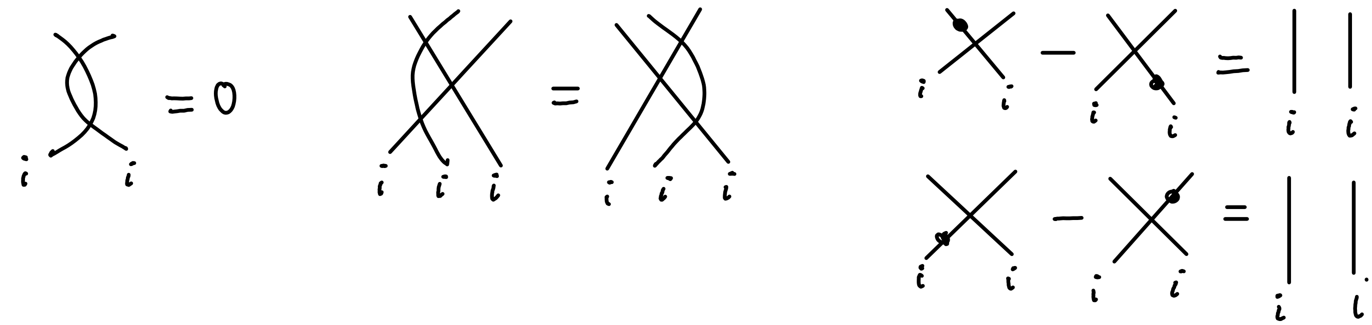
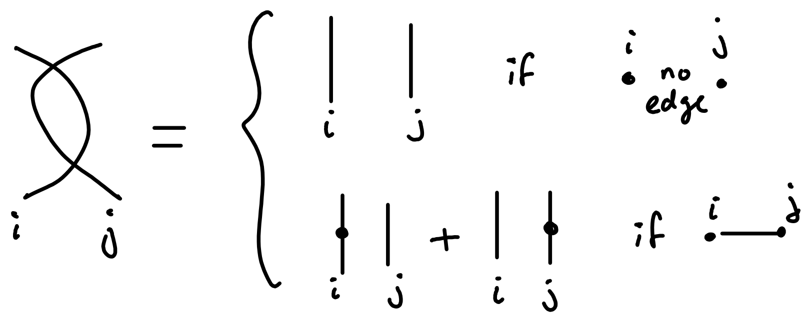
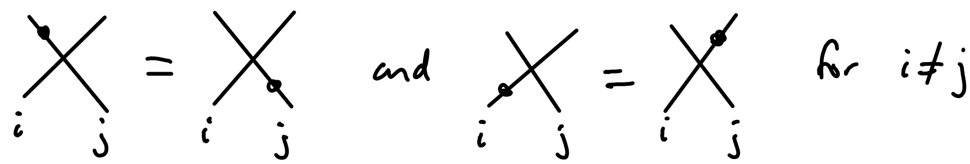
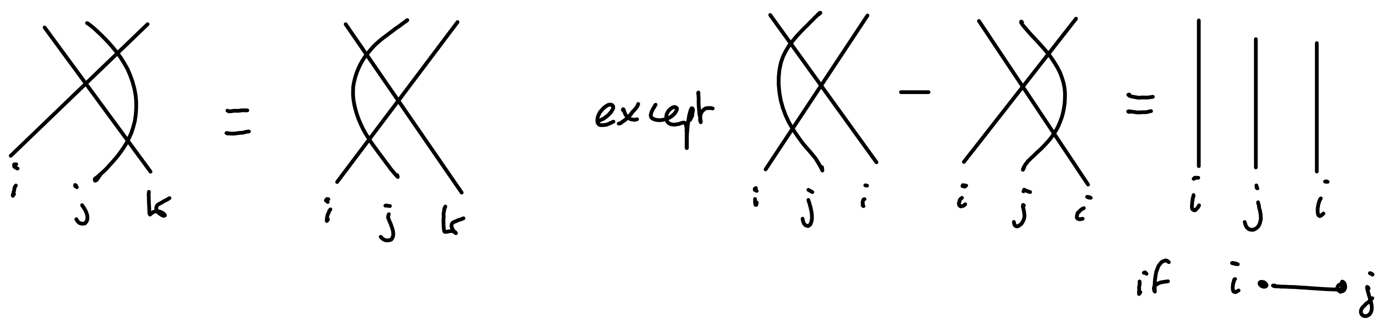
M. Khovanov – 29/5/08 – How to categorify 1/2 of a quantum group

Answer:

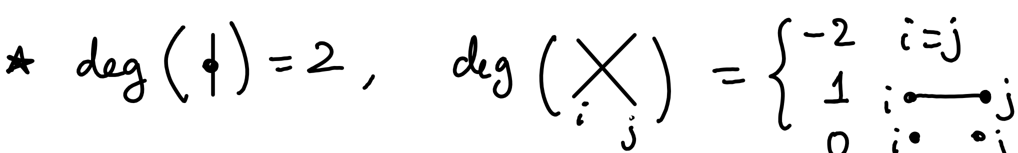






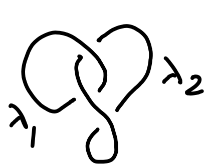


where $i, j, k \in I$ = set of vertices of Γ



$\mathfrak{g}$ simple Lie algebra, $U(\mathfrak{g})$ Hopf algebra $\rightsquigarrow$ $U_q(\mathfrak{g})$ quantum group

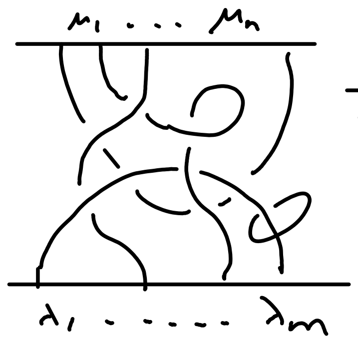
$\rightsquigarrow$ Reshetikhin-Turaev invariant:



links whose components are colored by positive integral weights of $\mathfrak{g}$ $\longmapsto$ $P(L, \lambda) \in \mathbb{Z}[q, q^{-1}]$

(positive integral weights $\longleftrightarrow$ finite dim. irreps of $U_q(\mathfrak{g})$)

We also get an <u>invariant of tangles</u>:



T = tangle with all components colored by positive integral weights

$$\longmapsto \qquad \begin{array}{c} V_{\mu_1} \otimes \cdots \otimes V_{\mu_n} \circlearrowleft U_q(\mathfrak{g}). \\ f(T) \Big\uparrow \quad \text{morphism of rep}^{\underline{\text{ns}}} \\ V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m} \circlearrowleft U_q(\mathfrak{g}) \end{array}$$

$$\left(x f(T) = f(T) x \quad \text{for } x \in U_q(\mathfrak{g}) \right)$$

- 3D $\rightsquigarrow$ 4D should enrich this to <u>categories</u>:

expect $\exists$ <u>triangulated</u> <u>categories</u> $\mathcal{C}_{\mu_1 \ldots \mu_n} \circlearrowleft \widetilde{U_q(\mathfrak{g})}$

$\mathcal{C}_{\lambda_1 \ldots \lambda_m} \circlearrowleft \widetilde{U_q(\mathfrak{g})}$

and <u>exact functor</u> $F(T)$

s.t. <u>grothendieck group</u> of $\mathcal{C}_{\lambda_1, \ldots \lambda_n}$ is $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$

and $[F(T)] = f(T)$ at level of Groth. groups.

- This picture is conjectural except simplest cases (eg. fund. rep$^{\underline{\text{ns}}}$ of $\mathfrak{sl}_n$)

[can't handle all weights λ!]

See also relation to sympl. geometry: Seidel-Smith-Manolescu

($\mathcal{C}$ = Fukaya category of a certain sympl. manifolds)

Let $\mathfrak{g} = n_+ \oplus \mathfrak{h} \oplus n_-$ (e.g. $\mathfrak{g} = sl_k$: $\mathfrak{h}$ = diagonal
$n_\pm$ = nilpotent $\begin{smallmatrix}\text{upper}\\ \text{lower}\end{smallmatrix}$ triangular)

$\rightsquigarrow \mathcal{U}(\mathfrak{g}) = \mathcal{U}(n^+) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(n^-)$

$\mathcal{U}(n^+) = \mathcal{U}^+$, build from it $\mathcal{U}_q^+$.

- generators of n_+: $E_i = \begin{array}{c} \\ i \end{array}\begin{pmatrix} & & i+1 \\ \cdots & \cdots & 1 \\ & & \end{pmatrix}$

satisfy $[E_i, [E_i, E_{i\pm1}]] = 0$

$[E_i, E_j] = 0$ for $|i-j| \geq 2$

$\rightsquigarrow$ encode this by a graph Γ with $\{\text{vertices}\} = I$,

$\overset{i}{\circ} \quad \overset{j}{\circ}$ means $[E_i, E_j] = 0$ i.e. $E_iE_j = E_jE_i$

$\overset{i}{\circ}\!\!-\!\!-\!\!-\!\!\overset{j}{\circ}$ means $[E_i, [E_i, E_j]] = 0$ i.e. $E_i^2E_j + E_jE_i^2 = 2E_iE_jE_i$

<u>Deform</u> the 2^{nd} relation to

$$E_i^2E_j + E_jE_i^2 = (q+q^{-1})\, E_iE_jE_i$$

or equivalently $E_i^{[2]}E_j + E_jE_i^{[2]} = E_iE_jE_i$

where $E_i^{[2]} = \dfrac{E_i^2}{[2]}$, $[2] = q + q^{-1}$

(in general $[k] = \dfrac{q^k - q^{-k}}{q - q^{-1}}$)

$\rightsquigarrow \mathcal{U}^+$ assoc. algebra over $\mathbb{Q}(q)$ gen^d by the E_i with these relations.

- define $\Delta E_i = E_i \otimes 1 + 1 \otimes E_i$ ($\rightsquigarrow$ bialgebra ...)

and define product on $\mathcal{U}^+ \otimes \mathcal{U}^+$: $(x_1 \otimes x_2)(x_1' \otimes x_2') = q^{|x_2||x_1'|} x_1x_1' \otimes x_2x_2'$

- $\mathcal{U}^+$ is multigraded: $\mathcal{U}^+ = \bigoplus_{\nu \in \mathbb{N}[I]} \mathcal{U}^+(\nu)$

$$\mathcal{U}^+(0) = \mathbb{Q}(q).1$$

- $\exists$ twisted bialgebra $\mathcal{U}^+_{\mathbb{Z}} \subset \mathcal{U}^+$

  = subalgebra $/ \mathbb{Z}[q, q^{-1}]$ generated by $E_i^{(n)} = \dfrac{E_i^{\,n}}{[n]}$

$$\mathcal{U}^+_{\mathbb{Z}} = \bigoplus_{\nu \in \mathbb{N}[I]} \mathcal{U}^+_{\mathbb{Z}}(\nu) \simeq \bigoplus_{\nu} K_0(R(\nu))$$

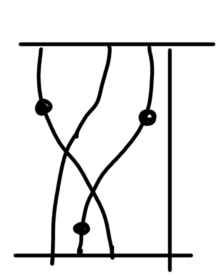
here for a ring A, $K_0(A)$ = ab. group gen^{d} by symbols of f.g. proj. modules $P \rightsquigarrow [P]$ w/ relations $[P \oplus Q] = [P] + [Q]$

& for a <u>graded</u> ring A, $K_0(A\text{-gpmod})$ (a $\mathbb{Z}[q, q^{-1}]$-module) = same construction w/ graded proj. A-modules

$\exists$ finitely many indecomposable projective modules $P_1 \dots P_m$ $\rightsquigarrow [P_1], .., [P_m]$ basis of K_0 as free $\mathbb{Z}[q, q^{-1}]$-module

Denote $[P\{k\}] = q^k [P]$.

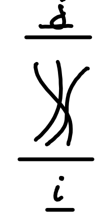
- $R(\nu)$ is spanned by braid-like diagrams mod relations listed at beginning.



$i_1 \dots i_m$

- <u>Namely</u>: given $\nu \in \mathbb{N}[I]$ (= linear combination of vertices of Γ)
  let $\mathrm{seq}(\nu) = \{$ all sequences of vertices of Γ with weight $\nu \}$
  e.g. $\mathrm{seq}(2i+j) = \{ iij, iji, jii \}$

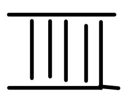
Then $R(\nu) := \bigoplus_{\underline{i},\underline{j} \in \mathrm{Seq}(\nu)} {}_{\underline{j}}R(\nu)_{\underline{i}}$

where ${}_{\underline{j}}R(\nu)_{\underline{i}}$ = spanned by diagrams 

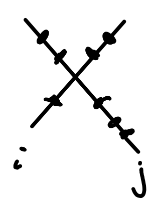
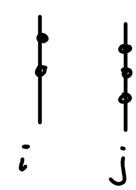
$R(\nu)$ is a ring, or equivalently a category
with objects $\mathrm{Ob}\,\mathcal{R}(\nu) = \mathrm{Seq}(\nu)$

product (composition) = concatenation of diagrams
(if compatible sequences, 0 otherwise)

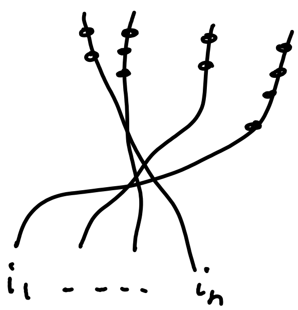
— in fact it's a <u>graded</u> ring, grading given at top.

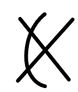
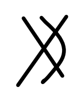
(<u>NB</u>: $\mathrm{id}_{\underline{i}}$ = $\underline{i}$  $\underline{i}$ trivial diagram)

Think of $\underline{R(\nu) = \text{CATEGORIFICATION OF } \mathcal{U}^+_{\mathbb{Z}}(\nu)}$

<u>Examples</u>:
- $R(0) = \mathbb{k}$ (empty diagram)
- $R(i) = \mathbb{k}[t]$ where generator = 
- $R(i+j)$ contains e.g.  and 

$R(\nu)$ has a spanning set (<u>basis</u>) given by permutn. diagrams



where
- all dots are at top
  [use relations to move them up!]
- any 2 strands cross at most once
  [use relations to simplify]
- make choice  vs 

Let's check this is correct... can build a map

$$\mathcal{U}^+_{\mathbb{Z}} \longrightarrow \bigoplus_{\nu} K_0(R(\nu))$$
$$E_{i_1} \cdots E_{i_m} \longmapsto [P_{\underline{i}}]$$

where $P_{\underline{i}} = \bigoplus_{\underline{j}} {}_{\underline{j}}R(\nu)_{\underline{i}}$ left projective $R(\nu)$-module

(similarly, ${}_{\underline{j}}P = \bigoplus_{\underline{i}} {}_{\underline{j}}R(\nu)_{\underline{i}}$ right proj. $R(\nu)$-module)

$$R(\nu) = \bigoplus_{\underline{i}} P_{\underline{i}}$$

- Also, $E_i^{(n)} = \dfrac{E_i^n}{[n]} \rightsquigarrow ??$

<u>Need</u>: relations in $R^{(\nu)}$ $\Rightarrow$ this constr$^{\underline{n}}$ is compatible with relation

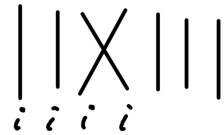
$$E_i^{(2)}E_j + E_j E_i^{(2)} = E_i E_j E_i$$

<u>Observe</u>: $R(ni) \simeq \mathrm{Mat}(n!, Z)$ ← center, where

$$Z(R(ni)) \simeq k[x_1, \ldots, x_n]^{S_n}$$

(use: nilHecke action on $k[x_1, \ldots x_n]$

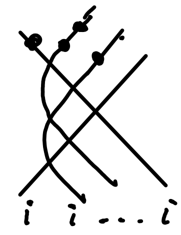
$= x_j$,



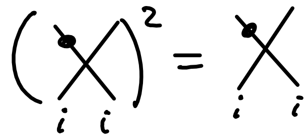
$= \partial_k$, $\partial_k f = \dfrac{f - {}^{s_k}f}{x_k - x_{k+1}}$

where ${}^{s_k}f$ = action of transposition $s_k = (k, k+1)$ by subst. variables $x_k \leftrightarrow x_{k+1}$)

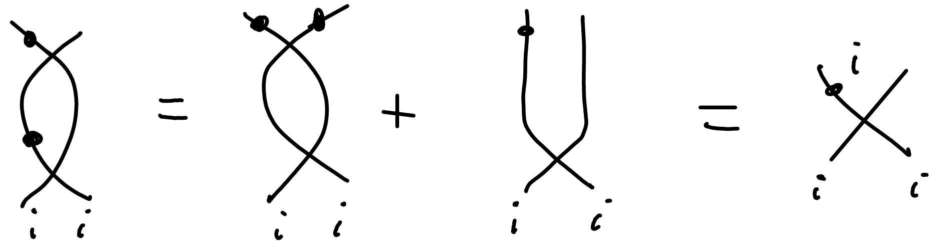
Idempotents $e^2 = e$: $\exists$ <u>$n!$ idempotents</u> which look like

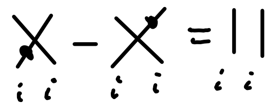


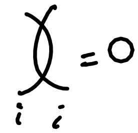
Example:



since



use 

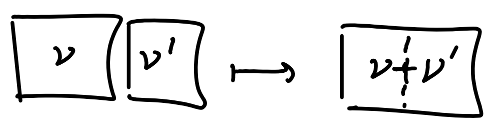
use 

Now map $E_i^{(n)}$ to Σ [idempotents].

- $\mathcal{U}^+_{\mathbb{Z}} = \bigoplus_{\nu \in \mathbb{N}[I]} \mathcal{U}^+_{\mathbb{Z}}(\nu) \cong \bigoplus_{\nu} K_0(R(\nu)) = K_0(R)$

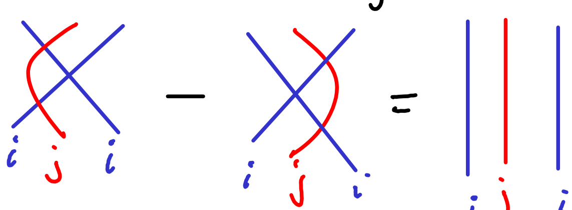
where $R = \bigoplus R(\nu)$, using $R(\nu) \otimes R(\nu') \longrightarrow R(\nu+\nu')$

side by side juxtaposition



- Now, check relation $E_j E_i^{(2)} + E_i^{(2)} E_j = E_i E_j E_i$:

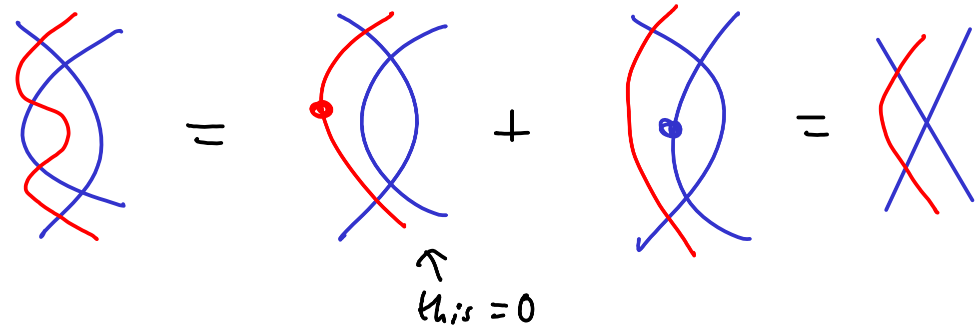
$P_{ji^{(2)}}$ $\quad$ $P_{i^{(2)}j}$



idempotent e_1 $\quad$ idempotent e_2

$e_1 + e_2 = 1_{iji}$, $\quad e_1 e_2 = e_2 e_1 = 0$

but relations ⇒

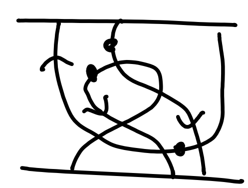
$P_{iji} =$ 

$P_{iji} \simeq P_{j i^{(2)}} \oplus P_{i^{(2)} j}$ ✓

<u>Further directions</u>:

★ To categorify all of $\mathcal{U}$, not just $\mathcal{U}^+$:
morally, "$\mathcal{U}^- \leftrightarrow$ rotate picture by 90°"

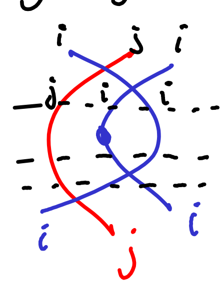
⇒ Should also allow diagrams where strands don't go up ....



★ <u>Conj</u>: using Wehrheim-Woodward, one should be able to see this geometrically:

$\mathcal{M}(\lambda, \nu)$ sympl. mfld with $H^{mid}(\mathcal{M}(\lambda,\nu)) \simeq V_\lambda(\nu)$

then should categorify to $\mathrm{Fuk}(\mathcal{M}(\lambda,\nu))$

Then cut diagrams 

into stages

⇝ each stage should yield a functor b/w Fukaya categories of relevant $\mathcal{M}$'s.